# Digital Image Processing

Kenneth R. Castleman

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Presentation by S. Wang



# Part Two - 1

## Linear Systems
## Fourier Transform



# Chapter 9

## Linear System Theory



## Introduction

- Questions to be discussed in this part:
  - Sampling effects
  - Spatial resolution
  - Linear filtering
- This chapter deals with analytical tools for solving the above problems
- The basis: the theory of linear systems.

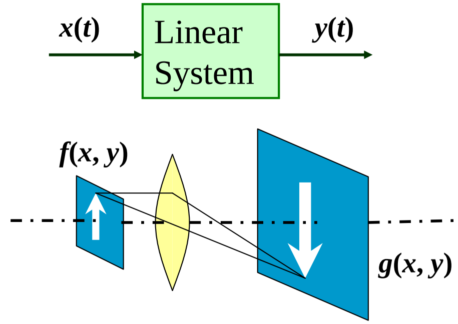
# Elementary Problems

- What is a linear system?
- What is a shift invariant system?
- Does a linear system generate new frequency contents?
- Does a linear system change waveform of an input signal?
- What effects does a linear system have on an input sinusoidal signal?
- When does a system of linear equations have
  - a set of unique solutions,
  - an infinite number of solution sets, or
  - no solution?



# Definition of System

- A system is a mapping, or a function, that defines relation between an input and the corresponding output.
- A system can be one-dimensional, two-dimensional, or high-dimensional.
- Consider one-dimensional functions of time first, and then generalize to two-dimensional functions of spatial variables.




1D system

2D system



# Linear System — Principle of Superposition

- Linearity: Supposing

$$x_1(t) \Rightarrow y_1(t)$$
$$x_2(t) \Rightarrow y_2(t)$$

The system is linear iff

$$x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$$

- Corollary (*a* is an integer):

$$ax_1(t) \Rightarrow ay_1(t)$$

- Axiom: ***a*** may also be irrational.



# Shift Invariance

- A system $x(t) \Rightarrow y(t)$
  is said to be shift-invariant if $x(t-T) \Rightarrow y(t-T)$
- For a shift invariant system, delaying the input merely shifts the output by the same amount without changing the nature of the output signal.
- In images, spatially shifting an input image does not change the output image except for an identical shift.

$$f(x, y) \Rightarrow g(x, y)$$

$$f(x+\Delta x, y+\Delta y) \Rightarrow g(x+\Delta x, y+\Delta y)$$

## Shift-Invariant Linear Systems

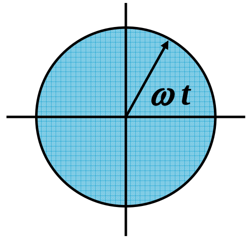
- Assumption of shift-invariance and linearity is good for many components of image processing systems:
  - Electrical networks,
  - Well-designed linear electronic networks, and
  - Optical systems.
- The next few chapters will mainly be concentrated on shift-invariant linear systems.



## Harmonic Signals & Complex Signals

- Physical signals are usually real:
  - voltage
  - sound pressure
  - pixel gray-level
- Real signals can be made complex for convenience.
- The following complex-valued signal can be viewed as a unit length vector rotating in the complex plane anti-clockwise at an angular speed $\omega = 2\pi f$.

$$\boxed{x(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t}$$






## Response to Harmonic Input

- Assume a harmonic input: $\boxed{x_1(t) = e^{j\omega t}}$
- The response of a shift-invariant linear system is

$$\boxed{y_1(t) = K(\omega, t)e^{j\omega t}}$$

**Find the property of $K(\omega, t)$**

where $\boxed{K(\omega, t) = y_1(t) / e^{j\omega t}}$

is a function of $\omega$ and $t$.

- Introducing time delay $T$ to $x_1(t)$ to generate another input:

$$\boxed{x_2(t) = x_1(t-T) = e^{j\omega(t-T)} = e^{-j\omega T}x_1(t)}$$



## Response to Harmonic Input (cont.)

- Response of the linear system to $x_2(t)$ is

$$\boxed{y_2(t) = K(\omega, t-T)e^{j\omega(t-T)} = \underline{K(\omega, t-T)}e^{-j\omega T}x_1(t)}$$

- An alternative way to find $y_2(t)$: Recalling

$$\boxed{ax_1(t) \Rightarrow ay_1(t)}$$

$$\boxed{x_2(t) = x_1(t)e^{-j\omega T} \Rightarrow y_1(t)e^{-j\omega T} = y_2(t) = \underline{K(\omega, t)}e^{-j\omega T}x_1(t)}$$

- Compare the above, the following must hold for any $T$:

$$\boxed{K(\omega, t-T) = K(\omega, t)}$$

- This can be true only when ***K* is independent of *t***:

$$\boxed{y(t) = K(\omega)x(t)}$$

## Response to Harmonic Input (cont.)

**Conclusions:**

- The response of a shift-invariant linear system to a harmonic input is that input multiplied by a frequency-dependent (**time-independent**) complex number.
- A harmonic input always produces a harmonic output at the same frequency.
- A linear system does not generate new frequency components, but changes magnitude and phase of harmonic components existing in the input signal.



## Harmonic Signals and Sinusoids

- Add another constraint to shift-invariant linear systems: **They preserve realness**.

**Real input** → **Shift-invariant linear system** → **Real output**

- Real and imaginary parts go through a shift-invariant linear system independently of each other.
- Add an imaginary part to a real signal, and let it go through a linear system. The real output can be extracted from the complex output. Analysis can be simplified as:

Sinusoid input → Harmonic input → **SILS** → Harmonic output → Sinusoid output



## Transfer Function

- The transfer function $K(\omega)$ is sufficient to specify the system. It contains all information about SILS.
- The polar form:

$$\boxed{K(\omega) = A(\omega)e^{j\phi(\omega)}}$$

where $A(\omega)$ is the **gain**, and $\phi(\omega)$ is the **phase shift**.

- Summary: 3 important properties of SILS:
  - Harmonic input $\Rightarrow$ harmonic output
  - Completely specified by the transfer function $K(\omega)$
  - The system causes **amplitude change** and **phase shift**.



## Convolution Operation

- To find a general expression of the output waveform $y(t)$ from a linear system — convolution integral may be obtained using the **superposition integral**:

Linearity property: $$y(t) = \int_{-\infty}^{\infty} f(t,\tau)x(\tau)d\tau$$

Shift invariance: $$y(t-T) = \int_{-\infty}^{\infty} f(t,\tau)x(\tau-T)d\tau$$

Add $T$ to $t$ and $\tau$: $$y(t) = \int_{-\infty}^{\infty} f(t+T,\tau+T)x(\tau)d\tau$$

$f$ only depends on $t-\tau$: ⇒ $$y(t) = \int_{-\infty}^{\infty} g(t-\tau)x(\tau)d\tau$$

## Convolution Operation (cont.)

- The convolution integral:

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)x(\tau)d\tau \overset{\Delta}{=} g(t) * x(t)$$

where $g(t)$ is the ***impulse response*** of the system, which must be ***real*** to maintain the realness of $y(t)$.

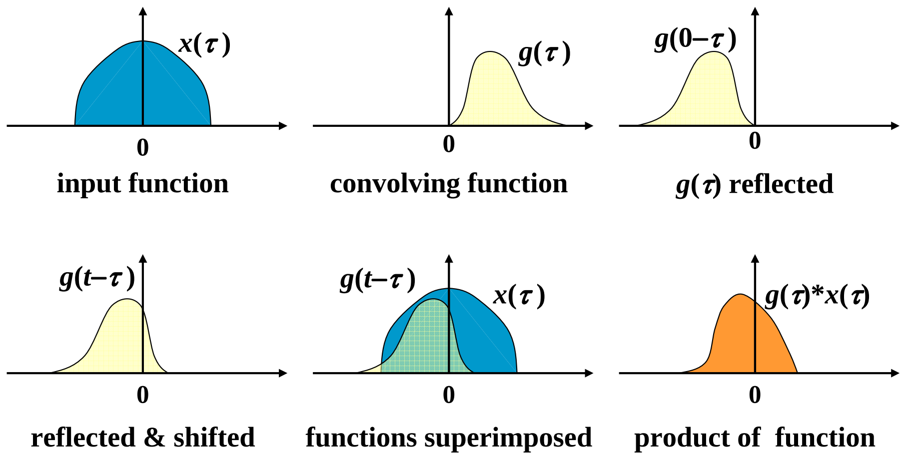
- There are two ways to specify input/output relationship:
  - Complex transfer function $K(\omega)$, and
  - Real impulse response $g(t)$.

$K(\omega)$ and $g(t)$ must be related.



## One Dimensional Convolution

- The process of 1D convolution:




Demo_conv1_cont



## Properties of Convolution

Commutative: $f * g = g * f$

Distributive: $f*(g+h) = f*g + f*h$

Associative: $f*(g*h) = (f*g)*h$

Differentiation:

$$\frac{d}{dt}[f*g] = f'*g = f*g'$$



## Discrete 1D Convolution

- Definition:

$$h(i) = f(i)*g(i) = \sum_j f(j)g(i-j)$$

- If lengths of ***f*** and ***g*** are ***n*** and ***m***, respectively, the length of ***h*** is ***N*=(*n*+*m*–1)**.
- Although discrete convolution and continuous convolution are different, they have many properties in common.
- Discrete convolution can be implemented on digital images, and is useful in image processing operations, especially in image enhancement and restoration.

Demo_conv1_disc

## Matrix Formulation of Convolution

- Discrete convolution in the form of summation:

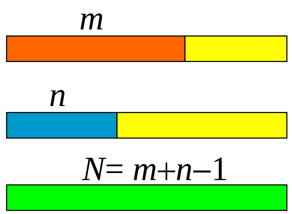
$$h(i) = f(i) * g(i) = \sum_j f(j)g(i-j)$$

Lengths of $f$ , $g$ and $h$ are $m$, $n$ and $N$=($n$+$m$–1), respectively.

- To take advantage of matrix operations, convolution must be re-formulated:
- First, zero-pad $f(i)$ and $g(i)$ to extend their length to $N$:

$$f_P(i) = \begin{cases} f(i) & 1 \le i \le m \\ 0 & m < i \le N \end{cases}$$

$$g_P(i) = \begin{cases} g(i) & 1 \le i \le n \\ 0 & n < i \le N \end{cases}$$






## Matrix Formulation of Convolution (cont.)

- Form an $N$×1 column vector **f** with elements $f_P(i)$, and a circulant matrix **G** whose rows are zero-padded $g_P(i)$ in a reversed order and its circularly right-shifted versions:

$$\mathbf{h} = \mathbf{G} \cdot \mathbf{f} \quad \text{or} \quad \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(N) \end{bmatrix} = \begin{bmatrix} g_P(1) & g_P(N) & \cdots & g_P(2) \\ g_P(2) & g_P(1) & \cdots & g_P(3) \\ \vdots & \vdots & \ddots & \vdots \\ g_P(N) & g_P(N-1) & \cdots & g_P(1) \end{bmatrix} \begin{bmatrix} f_P(1) \\ f_P(2) \\ \vdots \\ f_P(N) \end{bmatrix}$$

- The output is an $N$×1 column vector **h**.
- Each row of **G** contributes to one element in **h**.



## Two Dimensional Convolution

- For 2D continuous functions:

$$h(x,y) = f * g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)g(x-u, y-v)dudv$$

- For graphic illustration, see Fig 9-6 on p.153, where

$$f(x,y) = A\exp\left[-\frac{1}{2\sigma^2}(x^2+y^2)\right]$$

$$g(x,y) = \begin{cases} 1 & -1 \le x \le 1, -1 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

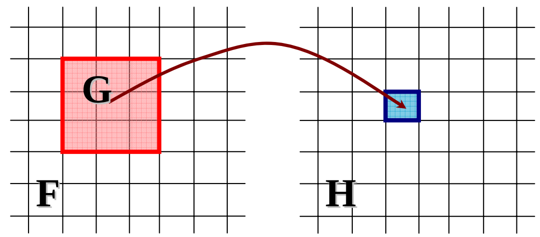
- Discrete 2D convolution: double integral becomes double summation:

$$\mathbf{H} = \mathbf{F} * \mathbf{G} \quad \text{or} \quad H(i,j) = \sum_m \sum_n F(m,n)G(i-m, j-n)$$



## Two Dimensional Convolution (cont.)

- Consider a 3×3 matrix **G**, called the ***kernel***, which is convolved with a larger image, **F**:




- Numbers of addition and multiplication operations:

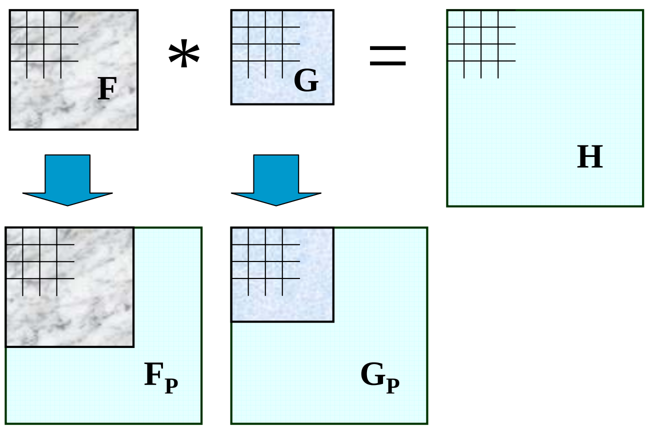
Number of elements in **G** × Number of pixels in **F**

- Border effect: extend image by (1) repeating the border, (2) filling in a constant, (3)periodically wrapping image, or (4) simply cutting borders to produce a smaller output.

Demo_conv2_m

## Matrix Formulation for 2D Convolution

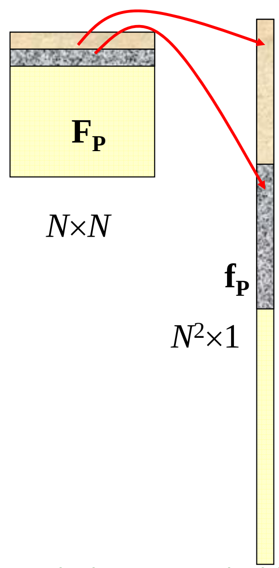
- As in the 1D case, **F** ($m_1 \times n_1$) and **G** ($m_2 \times n_2$) are extended to ($M \times N$) and made periodic, where $M \geq m_1 + m_2 - 1$ and $N \geq n_1 + n_2 - 1$. For simplicity, only consider $M = N$.






## Matrix Formulation (cont.)

- Form an $N^2 \times 1$ column vector $\mathbf{f}_P$ from $\mathbf{F}_P$ by row-stacking:






## Matrix Formulation (cont.)

- Each row of $\mathbf{G}_P$ forms $N \times N$ circulant matrix as in 1D case:

$$\boxed{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_i, \ldots, \mathbf{G}_N}$$

where

$$\mathbf{G}_i = \begin{bmatrix} g_i(1) & g_i(N) & \cdots & g_i(2) \\ g_i(2) & g_i(1) & \cdots & g_i(3) \\ \vdots & \vdots & \ddots & \vdots \\ g_i(N) & g_i(N-1) & \cdots & g_i(1) \end{bmatrix}$$

- Form a block circulant matrix, sized $N^2 \times N^2$:

$$\mathbf{G}_P = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_N & \cdots & \mathbf{G}_2 \\ \mathbf{G}_2 & \mathbf{G}_1 & \cdots & \mathbf{G}_3 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_N & \mathbf{G}_{N-1} & \cdots & \mathbf{G}_1 \end{bmatrix}$$

$$\Rightarrow \boxed{\mathbf{h}_P = \mathbf{G}_P \mathbf{f}_P}$$

**Remarks:**

– **Using matrix algebra, computation load is huge.**

– **Useful in filter design such as in image restoration.**



## Matrix Formulation: an Example

$$\mathbf{F} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \mathbf{F}_P = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \quad \mathbf{G}_P = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{h}_P = \mathbf{G}_P \cdot \mathbf{f}_P = \left[\begin{array}{ccc|ccc|ccc} -1 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 \\ \hline -2 & 0 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 1 & -1 \end{array}\right] \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -5 \\ -3 \\ 8 \\ -6 \\ -2 \\ 8 \end{bmatrix}$$

$$\mathbf{H} = \mathbf{F} * \mathbf{G} = \begin{bmatrix} -1 & -1 & 2 \\ -5 & -3 & 8 \\ -6 & -2 & 8 \end{bmatrix}$$

## Exercises: Discrete Convolution

- Assume $$\mathbf{F}=\begin{bmatrix}2 & 0 & 1\\ 3 & 2 & 2\end{bmatrix}\quad \mathbf{G}=\begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix}$$

  calculate **F**∗**G** by using the matrix representation.
- Calculate the above convolution using Matlab functions **conv** and **conv2**.
- Calculate **F**∗**G** using Matlab with different border treatments.

$$\mathbf{F}=\begin{bmatrix}1 & 1 & 2 & 3 & 3 & 2 & 1 & 0\\ 2 & 2 & 1 & 4 & 3 & 1 & 1 & 2\\ 1 & 2 & 5 & 5 & 4 & 1 & 2 & 2\\ 0 & 2 & 4 & 4 & 2 & 5 & 4 & 3\\ 0 & 2 & 2 & 3 & 4 & 6 & 5 & 3\\ 1 & 3 & 1 & 2 & 3 & 5 & 4 & 2\end{bmatrix}\quad \mathbf{G}=\begin{bmatrix}0 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 0\end{bmatrix}$$



## Applications of Convolution

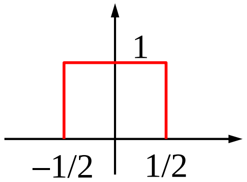
- **Filtering operations** such as smoothing, sharpening, etc.
- **Deconvolution**, removing the effects of imperfect linear systems that have operated on the image. For example:
  - Defects in the lens
  - Motion blurring
- **Noise removal or reduction**:
  - Estimating what was the signal before the noise was added.
  - Detecting presence of known features buried in noisy background.
  - Removing coherent (periodical) noise interference.
- **Feature Enhancement**: increase the contrast of specific features (edges, spots, etc.), usually at the expense of other objects in the image.



## Useful Functions in Linear System Theory

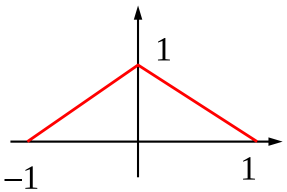
- Rectangular pulse: useful for modeling sampling windows and smoothing functions.

$$\Pi(x)=\begin{cases}1 & -1/2<x<1/2\\ 1/2 & x=\pm 1/2\\ 0 & \text{elsewhere}\end{cases}$$




- Triangular pulse: convolution of two rectangles.

$$\Lambda(x)=\Pi(x)*\Pi(x)=\begin{cases}1-|x|, & |x|\le 1\\ 0, & |x|>1\end{cases}$$






## Useful Functions (cont.)

- Gaussian Function: a smooth and unimodal function, useful for modeling sampling windows, display spots, etc.

$$f(x)=\exp\left[-\frac{x^2}{2\sigma^2}\right]$$

- When used as a probability density function, the area under the curve is normalized ($x_0$ is the mean):

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]$$

- An important property: Convolution of two Gaussians produces another Gaussian:

$$A\exp\left[-\frac{(x-a)^2}{2\sigma_1^2}\right]*B\exp\left[-\frac{(x-b)^2}{2\sigma_2^2}\right]=AB\exp\left[-\frac{x-(a+b)}{2(\sigma_1^2+\sigma_2^2)}\right]$$

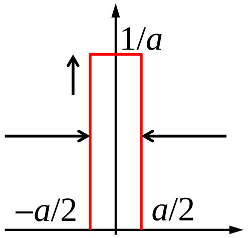
## Useful Functions (cont.)

- The impulse: Dirac delta function $\delta(x)$, a symbolic function defined by the following integral property:

$$\int_{-\infty}^{\infty}\delta(x)dx=\int_{-\varepsilon}^{\varepsilon}\delta(x)dx=1,\qquad \int_{-\infty}^{\infty}f(x)\,\delta(x)\,dx=f(0)$$

- The impulse can also be considered as the limit of a rectangular pulse with the width $\rightarrow$ 0:

$$\delta(x)=\lim_{a\to 0}\frac{1}{a}\Pi\left(\frac{x}{a}\right)$$




- Impulse is an identity function under convolution:

$$\delta(x)*f(x)=\int_{-\infty}^{\infty}\delta(\tau)f(x-\tau)d\tau=f(x-\tau)|_{\tau=0}=f(x)$$



## Useful Functions (cont.)

- Properties of the impulse function:
  - Shifting

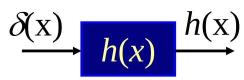
$$\int_{-\infty}^{\infty}f(x)\delta(x-x_0)dx=\int_{-\infty}^{\infty}f(x+x_0)\delta(x)dx=f(x_0)$$

  - Scaling

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

  - Output of a linear system to the impulse function is the system's impulse response. This explains the term **impulse response**.

$$\delta(x)*h(x)=\int_{-\infty}^{\infty}\delta(\tau)h(x-\tau)dx=h(x)$$






## Useful Functions (cont.)

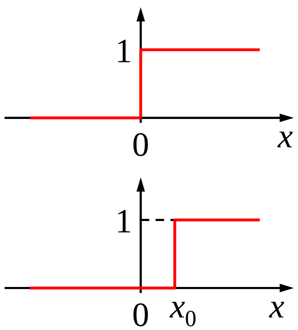
- Step function: derivative of delta function.

$$u(x)=\begin{cases}1 & x>0\\ 1/2 & x=0\\ 0 & x<0\end{cases}$$

$$u(x-x_0)=\begin{cases}1 & x>x_0\\ 1/2 & x=x_0\\ 0 & x<x_0\end{cases}$$

$$u'(x)=\delta(x)$$
$$u(x)=\int_{-\infty}^{\infty}\delta(x)dx$$

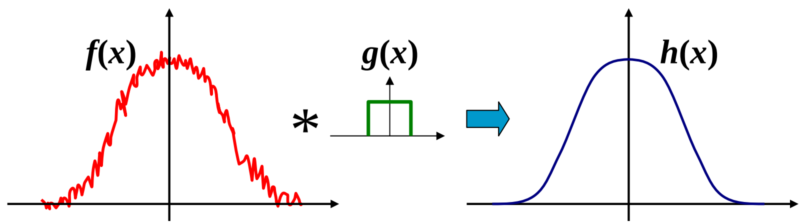
$$\int_{-\infty}^{\infty}u(x)f(x)dx=\int_{0}^{\infty}f(x)dx$$






## Convolution Filtering

- Smoothing — moving average




- Main feature of the convolution kernel for smoothing: **non-negativity**.
- The convolution window can be triangular or Gaussian: weighted moving average.

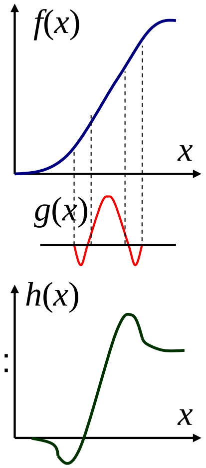
**Demo_conv1_m**

# Edge Enhancement: Sharpening

- A convolution kernel with a positive peak and negative side-lobes is a high-pass operator. The effects:
  - Increasing slope.
  - Producing overshoot (ringing).
- Example 2:

$$g(x) = 2\delta(x) - \exp\left(-x^2/2\sigma^2\right)$$

$$h = f * g = 2f(x) - f(x) * \exp\left(\frac{-x^2}{2\sigma^2}\right)$$




- Unsharp masking (old darkroom technique):

  **Original – Blurred image ⇒ Sharpened edge**

  Refer to Fig 9-18 on p.166.



# Summary

- Linear system: principle of superposition holds.
- Shift-invariance: system property independent of time.
- Harmonic signals: used to simplify analysis of linear systems.
- A linear system does not generate new frequency components.
- A shift-invariant linear system is specified by transfer function.



# Summary (cont.)

- Output waveform of a shift-invariant linear system is the convolution of the input with the impulse response.
- Convolution may be implemented to perform digital filtering.
- Convolution may be used for deconvolution, noise reduction, and feature enhancement.
- Convolving two Gaussians produces another Gaussian.
- The delta function is an identity function under convolution.



# Chapter 10

## Fourier Transform

## Introduction

- Fourier transform is a powerful tool in linear system analysis.
  - It is important to combine a theoretical knowledge of FT properties with a practical knowledge of their physical interpretation.
  - **Time spent to get familiar with the FT is well invested.**
- Two alternate ways to handle signal processing problems:
  - Time/spatial domain techniques
  - Frequency/transform domain techniques



## Introduction (cont.)

- It is useful to be able to think freely in either the spatial or the frequency domain.
- We first consider 1D problems, and then generalize to 2D.
- Laplace and Z-transforms will not be considered since image processing is not restricted to the requirement of **causality**.



## Causality

- A system is said to be causal, or physically realizable, if the output is caused by input of the past and present. **The output does not depend on the future input.**
- The impulse response of a causal system is one-sided: It must be zero for $t < 0$.
- In image processing, we are working with recorded data: Data in the **future** are available, or the present input can influence the past.
- As a result, convolution kernels can either be even or odd, and the coordinate origin can be chosen arbitrarily.



## The Continuous Fourier Transform

- One-dimensional Fourier transform pair:

$$\mathcal{F}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

$$\mathcal{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds$$

- Example: Gaussian function (for derivation, see p.173)

$$\exp[-\pi t^2] \quad \Leftrightarrow \quad \exp[-\pi s^2]$$

- Existence of FT: for many realistic signals, $f(t)$ either dies out rapidly for large $t$, or is truncated to a limited period or space, thus the absolute-integrability holds:

$$\int_{-\infty}^{\infty} | f(t) | dt < \infty$$

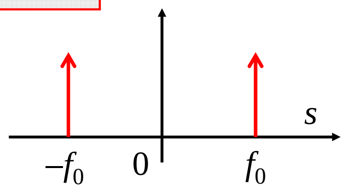
## Existence: Periodic & Constant Functions

- Strictly speaking, periodic functions such as **$f(t) = \cos\omega t$** and constant functions such as **$f(t) = 1$** do not satisfy the condition of absolute integrability.
- The problem can be handled with the help of delta function:

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{1}{2}[\delta(s+f_0)+\delta(s-f_0)]$$

$$\mathcal{F}\{\sin(2\pi f_0 t)\} = \frac{j}{2}[\delta(s+f_0)-\delta(s-f_0)]$$

$$\mathcal{F}\{1\} = \delta(s)$$






## Existence: Random Functions

- Random signals are aperiodic and do not die out with time, therefore are not absolutely integrable.
- Consider autocorrelation function of a random process:

$$R_f(\tau) = \mathrm{E}\{f(t)f(t+\tau)\} \underset{\text{For ergodic process}}{=} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)f(t+\tau)dt$$

- $R(\tau)$ is always real and even, and its Fourier transform is the power spectrum of the process.

$$P_f(\omega) = \int_{-\infty}^{\infty} R_f(\tau)\exp[-i\omega\tau]d\tau$$

**Remark: The existence of Fourier transform is not a problem for image processing.**



## The Discrete Fourier Transform

- View truncated function $g(t)$ as a cycle of periodic function.
- By making $s$ a discrete variable and taking integration only over $[-T/2, T/2]$, the FT of $g(t)$ becomes ($\Delta s = 1/T$)

$$G_n = G(n\Delta s) = \int_{-T/2}^{T/2} g(t)\exp[-j2\pi(n\,\Delta s\,t)]\,dt$$

- For a **periodic** signal, the frequency domain representation is **discrete**. $G(s)$ is non-zero only at equally spaced discrete intervals.

**Periodic in time domain ⇔ Discrete in frequency domain**

- The inverse transform:

$$g(t) = \sum_{n=0}^{\infty} G(n\Delta s)\exp[j2\pi(n\Delta s\,t)]\Delta s = \frac{1}{T}\sum_{n=0}^{\infty} G_n \exp\left[j2\pi\left(\frac{n}{T}t\right)\right]$$



## The Discrete Fourier Transform (cont.)

$$G_n = G(n\Delta s) = \int_{-T/2}^{T/2} g(t)\exp[-j2\pi(n\,\Delta s\,t)]\,dt$$

- Discretize both time and frequency, and the FT becomes

$$G_n = G(n\Delta s) = \sum_{i=-N/2}^{N/2} g(i\Delta t)\exp[-j2\pi(n\,\Delta s)i\Delta\,t]\,\Delta t$$

$$\Rightarrow \frac{T}{N}\sum_{i=-N/2}^{N/2} g_i \exp\left[-j2\pi\frac{n}{N}i\right] \quad \left(\because \Delta t = \frac{T}{N}, \Delta s = \frac{1}{T}\right)$$

$$g_i = g(i\Delta t) = \sum_{n=-\infty}^{\infty} G(n\Delta s)\exp[\,j2\pi(n\,\Delta s)i\Delta\,t]\,\Delta s$$

$$\Rightarrow \frac{1}{T}\sum_{n=-N/2}^{N/2} G_n \exp\left[j2\pi\frac{i}{N}n\right]$$

- Finally, range of summation becomes finite because

**Discrete in time domain ⇔ Periodic in frequency domain**

# The Discrete Fourier Transform (cont.)

- Changing range of indices, and making the forward and inverse transforms symmetric, DFT can be expressed as

$$F_n = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} f_i \exp\left[-j2\pi \frac{n}{N} i\right], \quad 0 \le n \le N-1$$

$$f_i = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} F_n \exp\left[j2\pi \frac{i}{N} n\right], \quad 0 \le i \le N-1$$

- DFT is closely related to CFT so that they can be viewed essentially equivalent in most image processing problems.
- We can use CFT when **analyzing** an image processing problem, and then **implement** the solution with DFT.

**Demo_CFT_DFT**



# The Fast Fourier Transform

- For DFT, number of multiplications and additions $\propto$ ***N*²**.
- Using the fast Fourier transform algorithm, the number of operations is reduced to the order of $\boldsymbol{N \log_2 N}$.
- Matrix form of DFT:

$$\begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{bmatrix} = \begin{bmatrix} W_{0,0} & W_{0,1} & \cdots & W_{0,N-1} \\ W_{1,0} & W_{1,1} & \cdots & W_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N-1,0} & W_{N-1,1} & \cdots & W_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}, \quad or\ \mathbf{F} = \mathcal{W}\,\mathbf{f}$$

- The FFT algorithm can be derived by considering the symmetry and periodicity of the terms in $\mathcal{W}$:

$$W_{n,i} = \frac{1}{\sqrt{N}} \exp\left[-j2\pi \frac{ni}{N}\right]$$



# FT of Some Useful Functions

| | | |
|---|---|---|
| $\exp(-\pi t^2)$ | $\Leftrightarrow$ | $\exp(-\pi s^2)$ |
| $\Pi(t)$ | $\Leftrightarrow$ | $\sin(\pi s)/(\pi s)$ |
| $\Lambda(t)$ | $\Leftrightarrow$ | $\sin^2(\pi s)/(\pi s)^2$ |
| $\delta(t)$ | $\Leftrightarrow$ | $1$ |
| $u(t)$ | $\Leftrightarrow$ | $1/2\,[\delta(s) - j/(\pi s)]$ |
| $\cos 2\pi ft$ | $\Leftrightarrow$ | $1/2[\delta(s+f)+\delta(s+f)]$ |
| $\sin 2\pi ft$ | $\Leftrightarrow$ | $j/2[\delta(s+f)-\delta(s+f)]$ |
| $\exp(2\pi ft)$ | $\Leftrightarrow$ | $\delta(s-f)$ |



# Properties of Fourier Transform

- Symmetry properties: in general, **complex** $\Leftrightarrow$ **complex**.
- Even and odd: a function can always be broken into even components and odd components:

$$f(t) = f_e(t) + f_o(t) \Leftrightarrow F(s) = F_e(s) + F_o(s)$$

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)] \Leftrightarrow F_e(s) = \int_{-\infty}^{\infty} f_e(t)\cos(2\pi st)\,dt$$

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)] \Leftrightarrow F_o(s) = -j\int_{-\infty}^{\infty} f_o(t)\sin(2\pi st)\,dt$$

- An even component produces an even component in FT, without introducing a *j*.
- An odd component produces an odd component in FT, introducing a *j*.

## Properties of Fourier Transform (cont.)

- Real and imaginary:
  - The real even part produces a real even part.
  - The real odd part produces an imaginary odd part.
  - The imaginary even part produces an imaginary even part.
  - The imaginary odd part produces a real odd part.

**For an even part, the realness is preserved.**

**For an odd part, the realness is flipped.**

- Of particular importance, the Fourier transform of a real signal is Hermitian (conjugate symmetric):

$$F(s) = F^*(-s)$$



## Evenness, Oddness and Symmetry

| *f*(*t*) | *F*(*s*) |
|---|---|
| Even | Even |
| Odd | Odd |
| Real and even | Real and even |
| Real and odd | Imaginary and odd |
| Imaginary and even | Imaginary and even |
| Imaginary and odd | Real and odd |
| Complex and even | Complex and even |
| Complex and odd | Complex and odd |
| Real | Hermite |
| Imaginary | Anti-Hermite |
| Hermite | Real |
| Anti-Hermite | Imaginary |



## Properties of FT: Addition Theorem

- Linearity of the Fourier transform:

$$\begin{array}{rcl} f(t) & \Leftrightarrow & F(s) \\ g(t) & \Leftrightarrow & G(s) \\ f(t)+g(t) & \Leftrightarrow & F(s)+G(s) \\ c\,f(t) & \Leftrightarrow & c\,F(s) \end{array}$$

- See Fig.10-1 on p.181 for an example.
- **Programming: plot waveform $x(t) = a \cos(2\pi ft)$ and its spectrum using proper axis scales, and study effects of truncation, windowing, and frequency changes.**



## Properties of FT (cont.)

- The shift theorem:

$$\mathcal{F}\{f(t-a)\} = \exp(-j2\pi as)F(s)$$

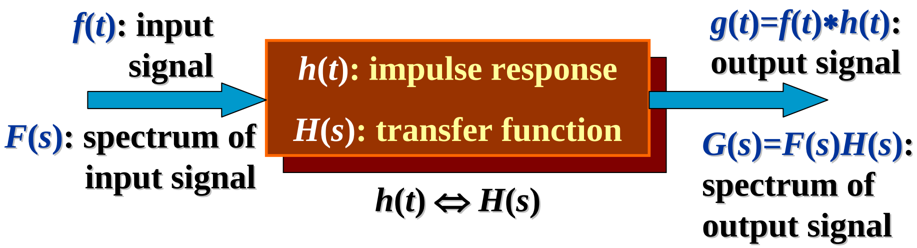
The exponent represents rotation in the transform domain — The amplitude is unchanged, while the phase redistributed.

- Convolution theorem: $f(t) * h(t) \Leftrightarrow F(s)H(s)$
- Scaling property: $f(at) \Leftrightarrow \dfrac{1}{|a|} F\left(\dfrac{s}{a}\right)$

  See Fig.10-2 on p.184
- Parseval's theorem: $\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$

## Linear Systems and Fourier Transform

- Terminology:




- **Linear system identification**: find the impulse response of a system given input and output signals.

$$\boxed{G(s)=F(s)H(s)\Rightarrow H(s)=G(s)/F(s),\quad F(s)\neq 0}$$

therefore, $$\boxed{h(t)=\mathcal{F}^{-1}\left\{\frac{G(s)}{F(s)}\right\}=\mathcal{F}^{-1}\left\{\frac{\mathcal{F}[g(t)]}{\mathcal{F}[f(t)]}\right\}}$$



## System Identification (cont.)

- Example 1: When $f(t) = \delta(t)$, the impulse response is identical to the output signal: $h(t)=g(t)$.
- Example 2: When input signal is $\Pi(t)$ and output is $\Lambda(t)$, the impulse response is also $\Pi(t)$. See Fig.10-4, p. 187.

$$\boxed{F(s)=\frac{\sin(\pi s)}{(\pi s)}}$$

$$\boxed{G(s)=\frac{\sin^2(\pi s)}{(\pi s)^2}}$$

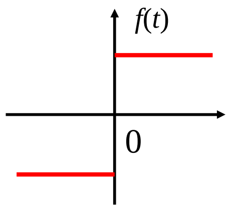
$$\boxed{h(t)=\mathcal{F}^{-1}\left\{\frac{\sin^2(\pi s)}{(\pi s)^2}\Big/\frac{\sin(\pi s)}{(\pi s)}\right\}=\mathcal{F}^{-1}\left\{\frac{\sin(\pi s)}{(\pi s)}\right\}=\Pi(t)}$$



## System Identification (cont.)

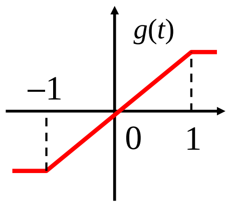
- Example 3: given

$$\boxed{f(t)=u(t)-\frac{1}{2}=\begin{cases}-1/2 & t<0\\ 0 & t=0\\ 1/2 & t>0\end{cases}\Leftrightarrow\quad F(s)=\frac{-j}{2\pi s}}$$




and

$$\boxed{g(t)=\begin{cases}-1/2 & t<-1\\ t & -1\le t\le 1\\ 1/2 & t>1\end{cases}\Leftrightarrow\quad G(s)=\frac{-j\sin(\pi s)}{2(\pi s)^2}}$$




$$\boxed{H(s)=\frac{G(s)}{F(s)}=\frac{\sin(\pi s)}{\pi s}\quad\Rightarrow\quad h(t)=\Pi(t)}$$



## System Identification (cont.)

- In general cases, no analytical expressions for the input and output signals are available. The system identification problem must be solved numerically.

$$\boxed{H(s)=G(s)/F(s)}$$

- Zeros in the input spectrum should be avoided.
- If both $F(s)$ and $G(s)$ have zeros at the same frequencies, obtain $H(s)$ at these frequencies by interpolation.

## Sinusoidal Decomposition

- (This section is for self-study)



## Some Concluding Remarks

- A linear system can change the amplitude and phase of a sinusoidal input, but cannot change the frequency.
- A linear system cannot generate new frequency contents.
- Two ways of viewing operation of linear system:
  - **Convolution with functions being reflected, shifted, multiplied, and integrated.**
  - **Sinusoidal decomposition followed by multiplication and re-summation.**
- Evenness and oddness are preserved between the two domains.
- In the Fourier domain, the negative frequencies are redundant, but convenient for mathematical operations.



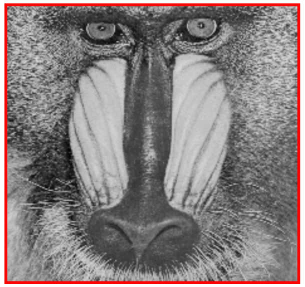
## Fourier Transform in Two Dimensions

- Definition:

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]\,dxdy$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[j2\pi(ux+vy)]\,dudv$$

where *u* and *v* are spatial frequencies.



**More examples**



## Discrete 2D Fourier Transform

- Definition:

$$G(m,n)=\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1} g(i,k)\exp\left[-j2\pi\left(m\frac{i}{N}+n\frac{k}{N}\right)\right],\quad \begin{matrix} m=0,1,\ldots,N-1 \\ n=0,1,\ldots,N-1 \end{matrix}$$

$$g(i,k)=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} G(m,n)\exp\left[j2\pi\left(i\frac{m}{N}+k\frac{n}{N}\right)\right],\quad \begin{matrix} i=0,1,\ldots,N-1 \\ k=0,1,\ldots,N-1 \end{matrix}$$

- A 2D DFT can be separated into two 1D DFT (row-wise and column-wise):

$$G(m,n)=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}\left\{\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} g(i,k)\exp\left[-j2\pi\left(n\frac{k}{N}\right)\right]\right\}\exp\left[-j2\pi\left(m\frac{i}{N}\right)\right]$$

## Fourier Basis Functions

- The 2D Fourier transform:

$$G(m,n)=\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1}g(i,k)A(i,k;m,n)$$

where $A(i, k; m, n)$ are the basis functions:

$$A(i,k;m,n)=\exp\left[-j\frac{2\pi}{N}(im+kn)\right]$$
$$=\cos\left[-\frac{2\pi}{N}(im+kn)\right]+j\sin\left[-\frac{2\pi}{N}(im+kn)\right]$$

- 2D basis functions are outer products of 1D basis.
- Any image can be viewed as a linear combination of basis functions.

Demo_2DFT



## Matrix Formulation of 2D DFT

- It can be shown that the DFT can be expressed in the following matrix notation:

$\mathbf{G}=\mathbf{F}\,\mathbf{g}\,\mathbf{F}$ where $\mathbf{F}$ is a unitary and symmetrical matrix:

$$\mathbf{F}=[f_{ik}]=\left[\frac{1}{\sqrt{N}}\exp\left(-j2\pi\frac{ik}{N}\right)\right]=\frac{1}{\sqrt{N}}\begin{bmatrix}W^0 & W^0 & \cdots & W^0\\ W^0 & W^1 & \cdots & W^{N-1}\\ \vdots & \vdots & \ddots & \vdots\\ W^0 & W^{N-1} & \cdots & W^{(N-1)^2}\end{bmatrix}$$

where $W=\exp[-j2\pi/N]$

- Unitary transform:

$\mathbf{F}^{-1}=\mathbf{F}^{*\mathrm{T}}$



## Properties of 2D Fourier Transform

**The following properties are unique to 2D FT:**

- Separability:

$$\text{If } f(x,y)=f_1(x)f_2(y)\ \Rightarrow\ F(u,v)=F_1(u)F_2(v)$$

- Similarity (scaling property):

$$\mathcal{F}\{f(a_1x+b_1y,\,a_2x+b_2y)\}$$
$$=(A_1B_2+A_2B_1)F(A_1u+A_2v,\,B_1u+B_2v)$$

where

$$A_1=\frac{b_2}{a_1b_2-a_2b_1}\qquad B_1=\frac{-b_1}{a_1b_2-a_2b_1}$$
$$A_2=\frac{-a_2}{a_1b_2-a_2b_1}\qquad B_2=\frac{a_1}{a_1b_2-a_2b_1}$$



## Properties of 2D Fourier Transform (cont.)

- Rotation: The spectrum rotates by the same angle as the image. Using the similarity property, and with

$$a_1=\cos\theta,\quad b_1=\sin\theta,\quad a_2=-\sin\theta,\quad b_2=\cos\theta$$
$$\therefore A_1=a_1,\quad A_2=b_1,\quad B_1=a_2,\quad B_2=b_2$$

Therefore,

$$\mathcal{F}\{f(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta)\}$$
$$=F(u\cos\theta+v\sin\theta,-u\sin\theta+v\cos\theta)$$

- Projection: When an image is collapsed into a 1D function:

$$p(x)=\int_{-\infty}^{\infty}f(x,y)dy\quad\Leftrightarrow\quad P(u)=F(u,0)$$

Try to prove.

This can be generalized to an arbitrary direction.

## Properties of 2D Fourier Transform (cont.)

- Circular symmetry and Hankel transform:

Suppose $f(x,y) = f_r(r), \quad r^2 = x^2 + y^2$

$$
\begin{aligned}
F(u,v) &= \iint f(x,y)\exp[-j2\pi(ux+vy)]\,dxdy \\
&= \iint f_r(r)\exp[-j2\pi qr\cos(\theta-\phi)]\,r\,drd\theta \\
&= \int_0^{\infty} f_r(r)\left[\int_0^{2\pi} \exp[-j2\pi qr\cos\theta]\,d\theta\right] r\,dr \\
&= 2\pi\int_{-\infty}^{\infty} f_r(r)\,J_0(2\pi qr)\,r\,dr = F_r(q)
\end{aligned}
$$

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
u &= q\cos\phi \\
v &= q\sin\phi
\end{aligned}
$$

- Therefore, $F_r(q) = 2\pi\int_0^{\infty} f_r(r)J_0(2\pi qr)\,r\,dr$

$$f_r(q) = 2\pi\int_0^{\infty} F_r(q)J_0(2\pi qr)\,q\,dq$$

Hankel transform of zero-order: p.200

**Bessel functions**



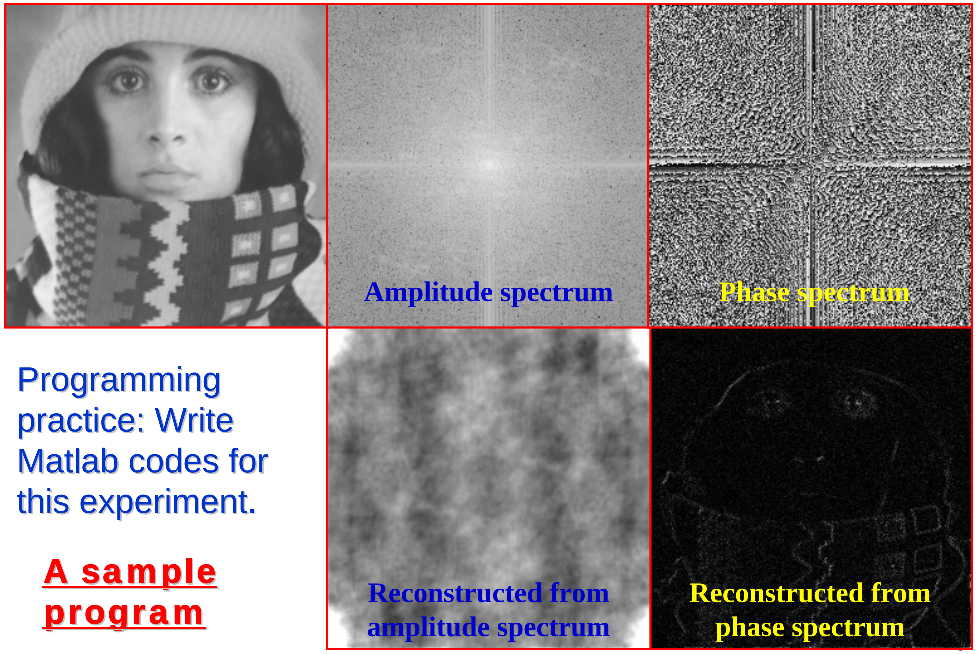
## Amplitude and Phase of Fourier Transform

- The amplitude specifies ***how much*** of each sinusoidal components is present.
- The phase information specifies ***where*** each of these components resides within the image.
- As long as components are kept in proper positions, the amplitudes appear to be less critical.
- Therefore, most practical filters affect amplitudes only while doing little or nothing to the phase in the spectrum.

**Demo_FFT2** **Demo_FFT_mag_ang**



## Amplitude and Phase of Fourier Transform




Programming practice: Write Matlab codes for this experiment.

**A sample program**



## Correlation and Power Spectrum

- Auto-correlation:

$$R_f(\tau) = f(t) * f(-t) = \int_{-\infty}^{\infty} f(t)f(t+\tau)dt$$

- Power spectrum: Fourier transform of $R_f(\tau)$:

$$
\begin{aligned}
P_f(s) &= \mathcal{F}\{R_f(\tau)\} = \mathcal{F}\{f(t) * f(-t)\} \\
&= F(s)F(-s) = F(s)F^*(s) = |F(s)|^2
\end{aligned}
$$

- Cross-correlation:

$$R_{fg}(\tau) = f(t) * g(-t) = \int_{-\infty}^{\infty} f(t)g(t+\tau)dt$$

- Cross power spectrum

$$P_{fg}(s) = \mathcal{F}\{R_{fg}(\tau)\}$$

## Summary of FT Properties

| Property | Time (space) domain | Frequency domain |
|---|---|---|
| Terminology | signal, impulse response | spectrum, transfer function |
| Definition | $\int_{-\infty}^{\infty} F(s)\exp[\ j2\pi st]ds$ | $\int_{-\infty}^{\infty} f(t)\exp[-j2\pi st]dt$ |
| Addition theorem | $af(x)+bg(x)$ | $aF(s)+bF(s)$ |
| Similarity theorem | $f(ax)$ | $(1/\|a\|)F(s/a)$ |
| Shift theorem | $f(x-a)$ | $\exp[-j2\pi as]F(s)$ |
| Convolution | $f(x)*g(x)$ | $F(s)G(s)$ |
| Differentiation | $df(x)/dx$ | $j2\pi s\,F(s)$ |
| Autocorrelation | $R_f(\tau)=f(x)*f^*(-x)$ | $\|F(s)\|^2=P_f(s)$ |
| Parseval theorem | $\int_{-\infty}^{\infty}\|f(x)\|^2dx=E$ | $\int_{-\infty}^{\infty}\|F(s)\|^2ds=E$ |
| Power theorem | $\int_{-\infty}^{\infty}f(x)g^*(x)dx=P$ | $\int_{-\infty}^{\infty}F(s)G^*(s)ds=P$ |



## Summary of Important Points

- FT establishes unique correspondence between a complex valued function of space and that of frequency.
- FT of Gaussian is another Gaussian.
- Evenness and oddness are preserved.
- FT of a real signal is Hermitian.
- FT is a linear transformation so that addition theorem holds.
- Shifting a waveform introduces phase shift in the spectrum.
- Convolution of two functions corresponds to multiplication of their spectra.
- Squeezing a signal broadens its spectrum, and vice versa.
- The energy of a signal is equal to that of its spectrum.
- Impulse response and transfer function are an FT pair.



## Summary of Important Points (cont.)

- The transfer function can be obtained from the input and output spectra.
- FT of a sinusoid is an impulse pair.
- Signal can be decomposed into infinite sum of sinusoids.
- A linear system operates separately on sinusoidal components of the input, and summing up at the output.
- If a function is separable, so is its FT.
- Rotating image rotates the spectrum by the same angle.
- Collapsing a 2D function onto a line *L* results in a profile of spectrum taken along a line in the same direction as *L*.
- The auto-correlation function and the power spectrum are a Fourier transform pair.



## Hermite and Anti-Hermite

- Hermite function:

$$\boxed{f(x)=f_e(x)+j\ f_o(x)}$$

$$\boxed{f(-x)=f_e(-x)+j\ f_o(-x)=f^*(x)}$$

- Anti-Hermite function:

$$\boxed{f(x)=f_o(x)+j\ f_e(x)}$$

$$\boxed{f(-x)=f_o(-x)+j\ f_e(-x)=-f^*(x)}$$

## Fourier Series

- A periodical signal can be expanded in to an infinite series:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(2\pi\frac{n}{T}t\right)+\sum_{n=1}^{\infty}b_n\sin\left(2\pi\frac{n}{T}t\right),\quad t\in\left[-\frac{T}{2},\frac{T}{2}\right]$$

where

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos\left(2\pi\frac{n}{T}t\right)dt$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin\left(2\pi\frac{n}{T}t\right)dt$$



## Zero-Order Bessel Function

$$J_0(z)=\frac{1}{2\pi}\int_0^{2\pi}\exp[-jz\cos\theta]\,d\theta$$



## The Schwartz Inequality

- It can be shown that

$$\int f^2(t)dt\int g^2(t)dt\ge\left[\int f(t)g(t)dt\right]^2 \qquad (1)$$

- First, define a nonnegative function of the variable $\lambda$:

$$Q(\lambda)=\int[\lambda f(t)+g(t)]^2dt\ge 0$$

- Expanding the integrand:

$$\int[\lambda f(t)+g(t)]^2dt=\lambda^2\int f^2(t)dt+2\lambda\int f(t)g(t)dt+\int g^2(t)dt \qquad (2)$$

- This is a quadratic form in the variable $\lambda$. The following inequality must hold to make (2) nonnegative.

$$\left[2\int f(t)g(t)dt\right]^2-4\int f^2(t)dt\int g^2(t)dt\le 0$$

- This proves (1).

$(b^2–4ac)<0 \Rightarrow$ The quadratic equation has no real roots.



## Waveform and Spectrum

```
T=1;N=512;
dt=T/(N-1);              % Sampling interval
fs=1/dt;                 % Sampling frequency
fn=fs/2;                 % Nyquist frequency
df=fs/N;
t=linspace(0,T,N);       % time axis
f1=linspace(0,fs-df,N);f2=linspace(-fn,fn-df,N);
freq1=df*25;freq2=df*56;
A=1;B=.8;
s=A*cos(2*pi*freq1*t)+B*sin(2*pi*freq2*t);
S1=abs(fft(s));S2=fftshift(S1);
subplot 311;plot(t,s);axis([0 T -2 2]);
subplot 312;plot(f1,S1);
axis([0 fs-df 0 max(S1(:))*1.2]);
subplot 313;plot(f2,S2);
axis([-fn fn-df 0 max(S2(:))*1.2]);
```

# 2D Fourier Transform: Magnitude & Phase

```
x=getimage;
[M,N,L]=size(x);
if L==3, d=rgb2ycbcr(x);x=d(:,:,1); end
X=fft2(double(x));
Xphase=exp(j*angle(X));Xmag=abs(X);     % Mag n phase
yphase=ifft2(Xphase);ymag=ifft2(Xmag); % reconstruct

a=5;b=800-M;
figure('Pos',[a b N M], 'Menu','none',...
   'Num','off','Name','original');
imshow(x);

a=a+min(N,300);
figure('Pos',[a b N M],'Menu','none',...
   'Number','off','Name','FFT: abs');
imshow(uint8(rescale(log(fftshift(Xmag+eps)))));
```



# 2D Fourier Transform (cont.)

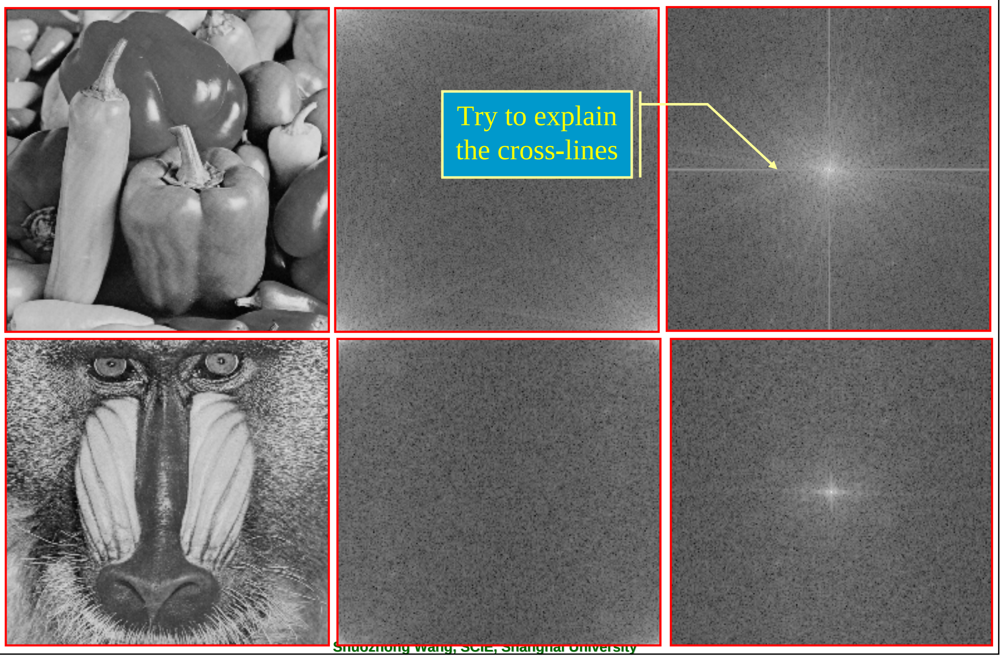
```
a=a+min(N,300);
figure('Pos',[a b N M],'Menu','none',...
   'Number','off','Name','abs-reconstructed');
imshow(uint8(clip(10*rescale(abs(ymag)),0,255)));

a=a+min(N,300);
figure('Pos',[a b N M],'Menu','none',...
   'Number','off','Name','FFT: angle');
imshow(uint8(rescale(log(fftshift(Xphase+eps)))));

a=a+min(N,300);
figure('Pos',[a b N M],'Menu','none',...
   'Number','off','Name','angle-reconstructed');
imshow(uint8(clip(5*rescale(abs(yphase)),0,255)));
```

**Go back**

Demo: testFFT

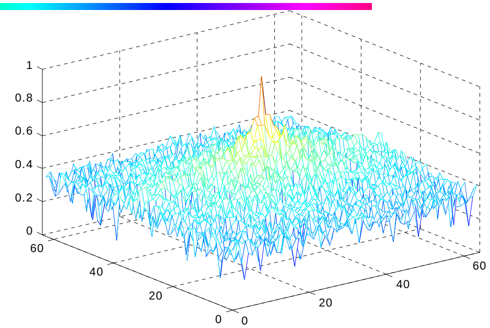


# Two-Dimensional DFT

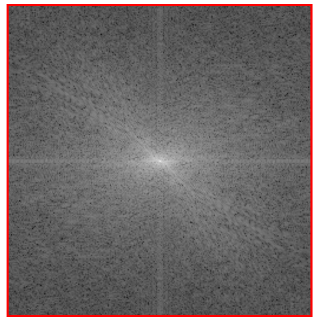





# FFT of Lena




**Go back**



## Eigenvalues and Eigenvectors

- For an $N \times N$ matrix **C**, there are $N$ scalars $\lambda_k$ ($k$ = 0, … , $N$–1) such that

$$\boxed{|\mathbf{C} - \lambda \mathbf{I}| = 0}$$

  $\lambda_k$ are called the **eigenvalues** of the matrix.
- The set of $N$ vectors $\mathbf{a}_k$ ($k$ = 0, … , $N$–1) such that

$$\boxed{\mathbf{C}\,\mathbf{a}_k = \lambda_k\,\mathbf{a}_k}$$

  are called the **eigenvectors** of the matrix **C**. They are $N \times 1$, and each corresponds to an eigenvalue.
- The eigenvectors form an **orthonormal** basis set.



## Orthogonal Basis

- Rows of **T** form an orthogonal basis for the $N$-dimensional vector space of all $N \times 1$ vectors.
- Example:

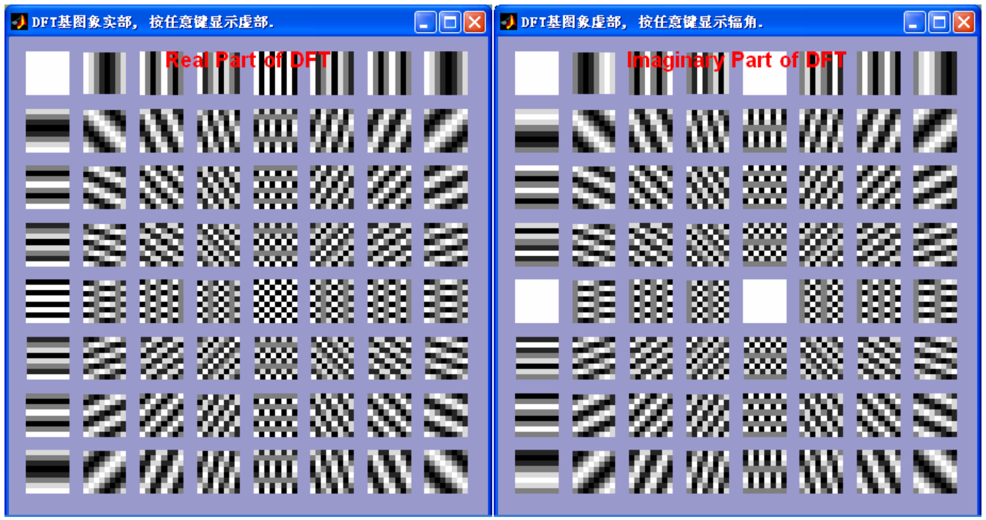
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}, \quad \text{if } \theta = \frac{\pi}{6}$$

- **The basis** — rows of **T**: $\mathbf{t}_1$=[0.866 –0.5], $\mathbf{t}_2$=[0.5 0.866].
- Any 2×1 vector can be expressed as a linear combination of $\mathbf{t}_1$ and $\mathbf{t}_2$.



## Fourier Basis Functions (Images)




Any 8×8 image can be viewed as a linear combination of these basis functions.