Notes on band pass filters

D.L.B. Jupp
October 2018

1. Background

A recurring theme in these seminars turns out to be band pass filters. That is a filter that allows energy associated with frequencies in a spectral band to pass into the output but stops or reduces energy associated with frequencies outside of the band. A band stop filter is simply the reverse of this. To illustrate the basic relationships between frequency and detail we will look at some examples of band pass filters and how they change images. However, since multi-resolution methods and wavelets as well as filter banks are all applications of band pass filters some extra discussion is best! The discussion summarises parts of Section 11.3 of Castleman.

2. Band pass filters

First there is the “ideal” band pass filter. This is illustrated in the frequency domain by symmetric bands as follows:

[Figure]

The symmetry of this frequency domain “filter” ensures the time or space domain filter is real. In the frequency domain it is defined very simply as:

$$G(s) = \begin{cases} 1 & f_1 \le |s| \le f_2 \\ 0 & otherwise \end{cases}$$

Castleman characterises this as a rectangular pulse convolved with an even impulse (delta-function) pulse pair. So:

$$\begin{aligned} & s_0 = (f_1 + f_2)/2 \\ & \Delta s = f_2 - f_1 \\ & G(s) = \Pi(s/|s|) * \left[\delta(s - s_0) + \delta(s - s_0)\right] \end{aligned}$$

The corresponding time domain filter is then obtained as:

$$g(t) = 2\Delta s \frac{\sin(\pi \Delta s t)}{\pi \Delta s t} \cos(2\pi s_0 t)$$

You can work this through if you wish! It can e plotted as in Castleman as:

[Fig]

Unfortunately, this filter is infinite in length and not very quickly reducing or compact. Applying this to a function with truncation produces a lot of "ringing". But the shape is something to keep in mind for later. The ideal band stop is dual to this filter.

3. A general band pass filter

Castleman then considers a more general shape. Rather than the rectangular pulse he suggests a non-negative (ie ≥ 0) unimodal function $K(s)$. It may look like the Figure below:

[Fig]

Then the pair of filters are:

$$G(s) = K(s) * \left[\delta(s - s_0) + \delta(s - s_0)\right]$$
$$g(t) = 2k(t)\cos(2\pi s_0 t)$$

So the result is again a cosine of frequency s_0 with an envelope $k(t)$. If K is a Gaussian function the result is much more tractable:

$$G(s) = Ae^{-s^2/2\alpha^2} * \left[\delta(s - s_0) + \delta(s - s_0)\right]$$
$$g(t) = \frac{2A}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2} \cos(2\pi s_0 t)$$
$$\sigma = \frac{1}{2\pi\alpha}$$

[Fig]

This pulse could be truncated with much less damage.

4. The difference of Gaussians filter

The Gaussian can illustrate a realisable filter for high pass enhancement and for a tractable band pass filter. The difference of Gaussians can be achieved by filtering data with two Gaussians and taking the one with the widest filter away from the one with the narrower width filter. In other work at NWU this was used as the basis for a filter bank.

Writing out the filters in the frequency and time domains:

$$G(s) = Ae^{-s^2/2\alpha_1^2} - Be^{-s^2/2\alpha_2^2} \qquad A \geq B, \alpha_1 \geq \alpha_2$$
$$g(t) = \frac{A}{\sqrt{2\pi\sigma_1^2}} e^{-t^2/2\sigma_1^2} - \frac{A}{\sqrt{2\pi\sigma_2^2}} e^{-t^2/2\sigma_2^2} \qquad \sigma_i = \frac{1}{2\pi\alpha_i}$$

The Figures illustrates the result in the frequency domain and the time domain.

[Figs]

If g1 is an impulse response the case is to subtract a Gaussian filtered image from the original image. This is illustrated in the following Figs:

[Fig]

That is, you are applying a high-pass filter. If, moreover, A=B then the low pass DC (trends) is removed. Previous work by NWU has used this to de-trend and allow Kriging to work without needed infinite variance.

In general G(0)=A1-A2 so as above, if A1=A2 then the DC is removed, meaning a high pass filter but if G(0)=1 then the low frequencies are preserved and the result is an enhancement of the higher frequencies. With careful choice, very high frequencies are also cut out. This type of filter will be seen among the “image reconstruction” filters. It is also said to “sharpen” an image or remove “blur”.

Finally, A1=A2 and both Gaussians are used, the result is a Gaussian band pass filter. Later we can look at how to estimate the actual width or effective width of the band.

5. Messages

Clearly, “ideal” band pass filters are hard to realise in the time domain. But it raises a question as to what does carrying out the filtering result in in the two domains? We will now look at this.

Gaussian band passes and filter banks will be encountered later. For now there is obviously a way to design the filters and estimate their effect that other functions may not provide. We have looked at the time domain to gain understanding but practical methods in Geography generally apply to images. The following will therefore use images and introduce the effects that appear when images are used rather than time series.