Notes on complex numbers

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[Information for this note was extracted from Ahlfors, L.V. (1966). *Complex Analysis*. McGraw Hill, 317p. and other freely available texts including internet posts as needed.]

1. Introduction

Some students may not have come across complex numbers or be a bit rusty. Complex numbers have played a fundamental role in wave theory, optics, electromagnetics and mathematical analysis of many kinds including potential theory and mapping.

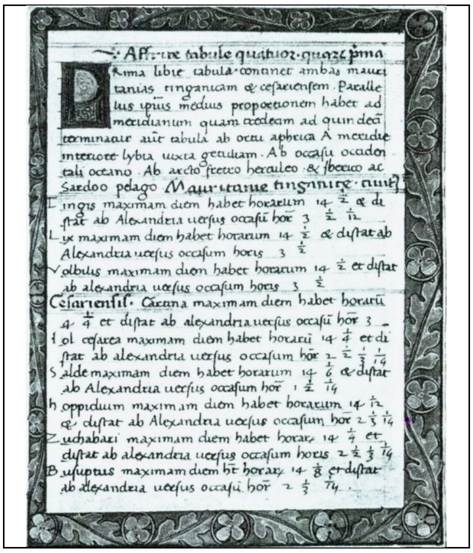
Numbers originally included cardinal (counting), ordinal (ordering) and the (finally) the integers (positive and negative numbers as well as a zero). Addition (+) and multiplication (x) of integers is well defined and include relationships such as:

$$
\begin{aligned}
&a+b=b+a \\
&a+0=a \\
&a\times b=b\times a \\
&a\times 1=a \\
&a\times(b+c)=a\times b+a\times c \\
&etc
\end{aligned}
$$

Division is possible but only in some cases which leads to studies of the highest common factor (HCF) and lowest common multiple (LCM) of integers and the study of prime numbers and the prime number factorisation of integers.

The equation $3x-5=0$ does not have an integer solution but rather its solution is something that became called a “fraction”, in this case 5/3, which cannot be simplified further as it could be if the top and bottom line had a common factor. The solutions as fractions leads to the study of all numbers that can be expressed as the ratio of two integers. They are called the “Rational” numbers or numbers formed by ratios.

This augmented number set was enough for many purposes in science and cartography. If you look at documents with calculations from (eg) Europe at about 1500 AD they will look something like this:



Affrice tabule quatuor quarum prima
Prima libie tabula continet ambas mauri
tanias tinganicam & cesariensem. Paralle
lus ipsius medius proportionem habet ad
meridianum quam sedecim ad quindecim
terminatur autem tabula ab ortu aphrica A meridie
interiore lybia iuxta getuliam. Ab occasu occiden
tali oceano. Ab arcto freto herculeo & iberico ac
Sardoo pelago Mauritanie tinganitane civitas
Tingis maximam diem habet horarum 14 1/2 & di
stat ab Alexandria versus occasum hor 3 1/2 1/12
Lyx maximam diem habet horarum 14 1/2 & distat ab
Alexandria versus occasum horis 3 1/2
Volbilis maximam diem habet horarum 14 1/2 et distat
ab alexandria versus occasum horis 3 1/2
Cesariensis. Cartenna maximam diem habet horarum
14 1/4 et distat ab alexandria versus occasum hor 3
Iol cesarea maximam diem habet horarum 14 1/4 et di
stat ab alexandria versus occasum hor 2 1/2 1/3 1/14
Salde maximam diem habet horarum 14 1/6 & distat
ab Alexandria versus occasum hor 1 1/2 1/14
h oppidum maximam diem habet horarum 14 1/12
& distat ab Alexandria versus occasum hor 2 1/3 1/14
Zuchabari maximam diem habet horarum 14 1/4 et
distat ab alexandria versus occasum horis 2 1/2 1/3 1/14
Busuptus maximam diem habet horarum 14 1/8 et distat
ab alexandria versus occasum hor 2 1/3 1/14.

There are only fractions or rational numbers in the calculations of latitudes and longitudes on this page. At that time integers and fractions (rational numbers) were the only way that numbers were represented and handled for calculations.

The integers are a part of the rational number system so there are more rational numbers than integers but it turns out that the rational numbers are still not "complete". For example, a problem arises when you need to find a square root. The equation $x^2 - 3 = 0$ has solution $x = \sqrt{3}$ but is $\sqrt{3}$ a rational number? This, it turns out, is NOT a rational number can only be computed approximately. The square root of any prime cannot be represented as the ratio of two integers and is called an "irrational" number. But the irrational numbers can be expressed as limits of series of rational numbers and the question arose what number system could include all of the rational numbers and irrational numbers? The answer was the "Real" number system (such as a floating point representation in a computer) in which all the limits of converging rational number series were included. Real numbers are what we think of a numbers and represent with the decimal point representation such as 4.7343…. and so on for as many places as needed. $\sqrt{3}$ needs an infinite number so can only be represented approximately – especially on a finite word length computer.

2. The need for complex numbers

Descartes defined Real numbers as number that could be represented on a line as being a given distance from the origin. They were the coordinates for his axes in the Cartesian system for representing vectors. Even geometry seemed complete at this stage and not need a serious update as was discussed above for the former coordinate systems.

However, there was still a problem to solve equations like:

$$P_2(x) = ax^2 + bx + c = 0$$

A solution of this equation is the "root" of the second order polynomial. Using the real number system almost all of the roots can be found, except in a particular situation. That is the solution can be answered algebraically as:

$$root = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Every case of such a polynomial therefore has a root (up to 2 roots) that is a real number *except* if $b^2 < 4ac$. In this case it was said, there was no solution. The solutions that existed were all part of Descartes' Real line.

At that point, complex numbers were defined and brought about a major advance in mathematics. The complex numbers enabled every equation like $P_2(x)$ to have 2 roots and (as it turns out) every equation like:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

To have n roots. Moreover, in the above polynomial, x and the coefficients ($a_k, k = 0, n$) may also be "complex numbers". The expansion of numbers into the enlarged complex system involves defining the primary complex number $j = \sqrt{-1}$. Alternatively, you can define j to be a new type of number such that $j^2 = -1$. The new "number" j is enough to define a whole new (and very useful) number system.

For example, consider the equation:

$$\begin{aligned} x^2 + x + 3 &= 0 \\ r &= \frac{-1 \pm \sqrt{1 - 4 \times 3}}{2} \\ &= \frac{-1 \pm j\sqrt{11}}{2} \end{aligned}$$

(Exercise: Substitute the solution into the equation and see if the solutions are the two roots of the polynomial).

3. Basic operations with complex numbers

A "complex" number is defined using pair of real numbers (a and b) and j written as:

$$\begin{aligned} z &= a + j \times b \quad or \\ z &= a + jb \end{aligned}$$

The real[1] numbers a and b are called the "real part" ($\operatorname{Re} z$) and "imaginary part" ($\operatorname{Im} z$) of the complex number. If b=0 the number z is said to be "purely real" and if a=0 the number z is called "purely imaginary". Zero is regarded as being both real and complex.

[1] If you substitute complex numbers for a and b, the equation simplifies to this form again so it is sufficient to define a and b as real numbers.

It can easily be found that the sum and products of complex numbers are also complex numbers:

$$(a+jb)+(c+jd)=(a+c)+j(b+d)$$
$$(a+jb)\times(c+jd)=(ac-bd)+j(ad+bc)$$

Remembering the polynomial above, substituting one root gives you:

$$\begin{aligned}&\left(\frac{-1+j\sqrt{11}}{2}\right)^2+\left(\frac{-1+j\sqrt{11}}{2}\right)+3\\&=\frac{1}{4}\left(-10-j\times2\sqrt{11}\right)+\frac{1}{2}\left(-1+j\sqrt{11}\right)+3\\&=0\end{aligned}$$

It is the same result for the other root so the polynomial has two complex roots.

There is an important and special complex number related to a given complex number z called its “complex conjugate” ($\overline{z}$) and there is another important real number associated with the complex number called its “modulus” defined as $|z|=\left(z\overline{z}\right)^{1/2}$.

That is, if $z=a+jb$:

$$\overline{z}=\text{conjugate}=a-jb$$
$$|z|=\sqrt{z\times\overline{z}}=\text{modulus}=(\text{a}^2+b^2)^{1/2}$$

Using the multiplication formulae:

$$\begin{aligned}z\times\overline{z}&=z\overline{z}\\&=(a+jb)(a-jb)\\&=a^2+b^2\\&=|z|^2\end{aligned}$$

It is easy to see that $\overline{\overline{z}}=z$. That is, the conjugate of the conjugate is the original complex number. What is less obvious but easily shown is that:

$$\text{Re}\ z=\frac{z+\overline{z}}{2}$$
$$\text{Im}\ z=\frac{z-\overline{z}}{2j}$$

And

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$$
$$\overline{z_1 z_2} = \overline{z}_1\, \overline{z}_2$$

These definitions can be used to determine the complex number which is the quotient (q) of two complex numbers z_1 and z_2:

$$\begin{aligned} q &= \frac{z_1}{z_2} = \frac{a + jb}{c + jd} \\ &= \frac{z_1 \overline{z}_2}{z_2 \overline{z}_2} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} \\ &= \frac{z_1 \overline{z}_2}{|z_2|^2} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \end{aligned}$$

The quotient is again a complex number in the standard form.

As a special case:

$$\frac{1}{z} = \frac{1}{a + jb} = \frac{a - jb}{a^2 + b^2}$$

So the reciprocal of a complex number is again a complex number.

Finally, we can look at the operation that provided a problem for the rational numbers - the square root. If $z = a + jb$, what is $\sqrt{z}$?

What we need are values x and y that solve the following equation:

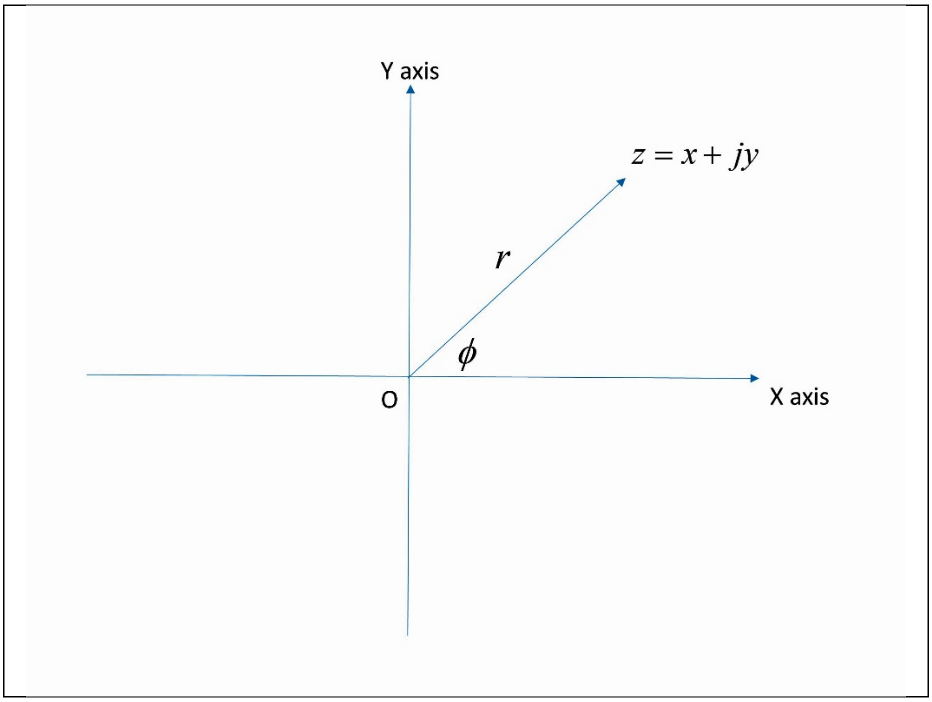
$$(x + jy)^2 = a + jb$$

It can be an exercise for students wishing to practice manipulating complex numbers! The solution is (Ahlfors, 1966):

$$\sqrt{z} = \sqrt{a + jb} = \pm\left(\sqrt{\frac{a + |z|}{2}} + j\frac{b}{|b|}\sqrt{\frac{-a + |z|}{2}}\right)$$

4. The complex plane

The complex numbers have an important geometrical interpretation that is especially useful when considering Laplace and Fourier transforms. In this representation the complex number $z = x + jy$ is considered to be a point in the Euclidean (x,y) plane.



The point in Cartesian coordinates where z is placed would normally be represented as (x,y). Using polar coordinates relative to the origin O, the values of the X and Y ordinates can be written as:

$$x = r\cos\phi$$
$$y = r\sin\phi$$

In this form, the radius (or modulus) (r) and angle (or argument, arg) (ϕ) are as shown on the Figure and lead to the representation of the complex number as:

$$z = r(\cos\phi + j\sin\phi)$$

That is, $|z| = r$ and the angle ϕ (or argument) traces out the circular variations of the complex number in the complex plane. At the four axis crossings, the points on the unit circle ($r = 1$) trace out the numbers $[1, j, -1, -j]$ indicating that 1 is like a "unit vector" for the X axis and j is like a "unit vector" for the Y axis.

By direct evaluation you can show that if two complex numbers are multiplied:

$$z_1 z_2 = r_1 r_2 \left(\cos(\phi_1 + \phi_2) + j\sin(\phi_1 + \phi_2)\right)$$
$$|z_1 z_2| = |z_1||z_2| = r_1 r_2$$
$$arg\ z_1 z_2 = arg\ z_1 + arg\ z_2$$

These and other relationships can be used to show that the n'th power (positive or negative) of a complex number can be written:

$$z^n = r^n \left(\cos(n\phi) + j\sin(n\phi)\right)$$

For the n'th root of a complex number the expression in Ahlfors (1966) resolves to:

$$z^{1/n} = \sqrt[n]{r}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + j\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right), \qquad k = 0,1,\ldots,n-1$$

Finally, the main summarising formula that will be needed for Fourier and Laplace transforms, called Euler's formula after its discoverer, is more difficult to prove and says:

$$e^{jz} = \cos(z) + j\sin(z)$$

In Euler's formula, z can be a complex number and there is a corresponding set of relationships between exponential and trigonometric formulae that are equivalent to this formula. For example:

$$\cos(z) = \frac{e^{jz} + e^{-jz}}{2}$$
$$\sin(z) = \frac{e^{jz} - e^{-jz}}{2j}$$

Euler's formula is fundamental to the use of Laplace and Fourier transforms in linear systems analysis. The formulae for cos and sin also appear when considering band pass and band reject filters. As a trivial example, the representation of a complex number in the complex plane can be written:

$$z = re^{j\phi}$$

This is the form of a harmonic input to a linear system that will be encountered in the course.

5. Discussion regarding Linear Systems

In Castleman section 9.2 he introduces the idea of a harmonic signal. It is a complex valued input signal of the form:

$$x(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

He describes it in terms of the complex plane representation of complex numbers as a unit vector rotating with angular velocity ω. This is related to frequency by $\omega = 2\pi f$. Castleman shows how for a shift-invariant linear system there is a function K that is independent of time such that the response of the linear system can be written:

$$y(t) = K(\omega)x(t)$$

The function K is the harmonic response of the shift-invariant linear system. In the course, some harmonic responses will be discussed as a prelude to Fourier transforms.

Using the complex plane representation of a complex number obtains the expression:

$$K(\omega) = A(\omega)e^{j\phi(\omega)}$$

The system response to the harmonic input in full is then:

$$\begin{aligned} y(t) &= K(\omega)e^{j\omega t} = A(\omega)e^{j\phi}e^{j\omega t} = A(\omega)e^{j(\omega t+\phi)} \\ &= A(\omega)\left(\cos(\omega t+\phi) + j\sin(\omega t+\phi)\right) \end{aligned}$$

In reality, inputs and outputs are usually not complex numbers. A DEM has real numbers, a time series of satellite images has real numbers and erosion from a slope is not usually expressed as a complex variable. Castleman described how the real valued response $\mathrm{Re}\,x(t) = \cos(\omega t)$ can be said to produce the real valued output $\mathrm{Re}\,y(t) = A(\omega)\cos(\omega t+\phi)$. He points out that in many cases the complex plane is used for intermediate derivations in this way. In the course, some cases are examined in which the purpose is simply to estimate $A(\omega)$ (modulation) and ϕ (phase shift) for the linear system when its input is the function $\cos(\omega t)$.

System response to a harmonic input leads naturally to Fourier transforms. In this case, the angular velocity is not used but frequency (cycles per sec) expressed as the variable "s". The forward Fourier integral is:

$$\Im\left\{f(t)\right\} = F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt$$

In this integral, t and f are usually real valued but F is a complex variable. If f is the inverse of a Fourier function F then it may also be complex. But is many cases, f is real.

Expanding the integral using Euler's formula:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt = \int_{-\infty}^{\infty} f(t)\left(\cos(2\pi st) - j\sin(2\pi st)\right)dt \\ &= \int_{-\infty}^{\infty} f(t)\cos(2\pi st)dt - j\int_{-\infty}^{\infty} f(t)\sin(2\pi st)dt \end{aligned}$$

$$\mathrm{Re}(F(s)) = \int_{-\infty}^{\infty} f(t)\cos(2\pi st)dt$$

$$\mathrm{Im}(F(s)) = -\int_{-\infty}^{\infty} f(t)\sin(2\pi st)dt$$

That is, the real and imaginary parts of F(s) can be separately computed by cosine and sine transformations. Sometimes these sine and cosine integrals are easier to compute than the exponential form and sometimes the exponential is easier. The formulae derived here based on the complex plane all have their uses.

In Castleman, he discusses odd and even functions as well as symmetries and their effect on the Fourier transform. These all depend on manipulating the different forms of the transformation. See if you can prove (using the various formulae from this added note) the statement in Castleman Chapter 10 Equation (36) that for a real input function (our normal case) that:

$$F(s) = \overline{F}(-s)$$ [2]

It is an important property!

6. Afterword

The Euler formula is very famous in history for many reasons. Among the more unusual reasons was that it is said by the philosopher Dieudonné Thiébault (although some others doubt it ever happened) that a French philosopher named Denis Diderot visited the court of Russia in 1773-4. He was an atheist and somewhat flamboyant in debate. He was asked to debate Leonhard Euler (who worked in Russia at the time) concerning a proof Euler had found for the existence of a supreme deity i.e. God. Diderot turned up to debate and Euler is said to have simply said:

"Monsieur, $e^{j\times\pi}+1=0$ donc Dieu existe, Repondez!"

Diderot had no answer and immediately left the court never to return. The court was overjoyed that the atheist had lost the debate. People feel Diderot knew no mathematics and was therefore fooled by the equation and had no way to answer ad so ran away. But Diderot was also a mathematician so maybe it was out of respect for the great Euler that he did not reply.

This equation is (of course) Euler's formula with $z=\pi$ and his assertion that it implied the existence of a God was that in one neat equation, 7 symbols being the primary complex number j, the transcendental numbers e and π, the fundamental operations $\times$ and $+$ and the fundamental numbers 0 and 1 were all linked in one equation. Possibly one should include equality (=) itself to make 8 symbols. This could be said by some to imply that there was a transcendental intelligence behind the structures of logic, mathematics and the universe.

Equally possible is that the old story is a fable and it never occurred at all! Many who have reported the event claimed the equation was a different one that made no sense at all. At least we know that Euler existed, that Euler's formula exists and we know for certain that Richard Feynman believed this to be the most remarkable equation in mathematics.

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[2] Equally this shows $F(-s) = \overline{F}(s)$.